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LETTER TO THE EDITOR

Exact solution of the Ising model on a 4–6 lattice

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Abstract
We consider an Ising model on a lattice of squares and hexagons (the '4–6' lattice), which does not appear to have been studied previously. The critical temperature is obtained as the root of a sixth-order equation. An exact expression is obtained for the spontaneous magnetization.

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In the years following Onsager's pioneering solution of the square lattice Ising model (Onsager 1944) and solutions for the other regular two-dimensional lattices (see, for example, Green and Hurst (1964)), exact solutions have also been obtained for a number of more complex lattices. We mention, in particular, the so-called 'Union Jack' lattice. (Vaks *et al* 1965, Lin and Wang 1987, Choy and Baxter 1987) and the 4–8 or 'bathroom tile' lattice (Utiyama 1951, Lin *et al* 1987, Baxter and Choy 1988). It is known, in fact, that all planar Ising models (i.e. with non-crossing bonds) are, in principle, solvable by the Pfaffion method (Green and Hurst 1964), and hence one could say that the problem is solved. However, it is still of interest to compute the critical point, free energy and magnetization explicitly for particular lattices.

The 4–8 lattice provides one example of tiling the plane with regular polygons of more than one kind. Other examples are easily constructed. Lin and Wang (1988) provide a solution for a 4–6 lattice, made of squares and hexagons, as a special case of the 4–8 lattice (figure 1(*a*)). We have identified a different 4–6 lattice (figure 1(*b*)) and the solution of this lattice is the subject of this letter.

There are at least two different approaches to the solution. The procedure we follow is to first transform the lattice to a square lattice with more complex interactions. This is followed by a transformation to an 8-vertex model. As the vertex weights satisfy the free-fermion condition (Fan and Wu 1970), the free energy and critical point are obtainable by standard methods. The magnetization is more difficult to obtain and is derived below.

Let us start by defining A sites (four-fold coordinated) and B sites (three-fold coordinated), and considering an elementary square of B sites and their connections to A sites, as shown in figure 2(*a*). We assume N A sites, which form a square lattice, and consequently have

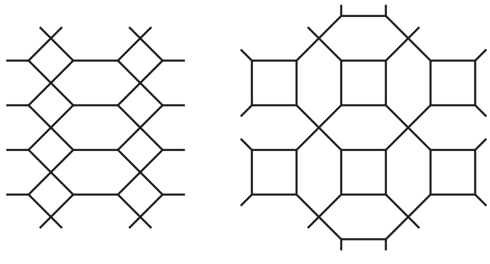

Figure 1. (*a*) The 4–6 lattice of Lin and Wang (1988). (*b*) The 4–6 lattice considered in this paper.

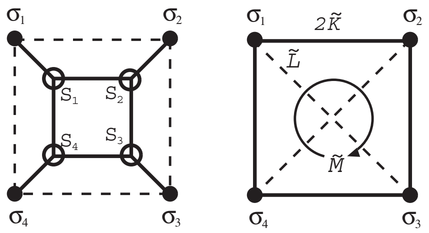

Figure 2. (*a*) An elementary A square. Summation over the internal B spins yields the generalized Ising model shown in (*b*).

$4N$ B sites. Further, we assume nearest-neighbour ferromagnetic interactions with coupling constant $K \equiv \beta J$, $J > 0$. It is straightforward to generalize this to the case of differing coupling constants K_1, K_2, but we do not pursue this.

The partition factor of an elementary square of A spins is then

$$Z(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \sum_{\{s\}} \exp[K(s_1\sigma_1 + s_2\sigma_2 + s_3\sigma_3 + s_4\sigma_4)] \exp[K(s_1 s_2 + s_2 s_3 + s_3 s_4 + s_1 s_4)] \tag{1}$$

and it is straightforward to carry out the summation over B spins and to express it in the form

$$Z(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \Lambda(K) \exp[\tilde{K}(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_1\sigma_4)] \\ \times \exp[\tilde{L}(\sigma_1\sigma_3 + \sigma_2\sigma_4)] \exp[\tilde{M}\sigma_1\sigma_2\sigma_3\sigma_4] \tag{2}$$

with

$$\Lambda e^{4\tilde{K}+2\tilde{L}+\tilde{M}} = e^{8K} + 4e^{2K} + 5 + 4e^{-2K} + 2e^{-4K} = P_1(K) \tag{3}$$

$$\Lambda e^{-4\tilde{K}+2\tilde{L}+\tilde{M}} = 2e^{4K} + 4e^{2K} + 5 + 4e^{-2K} + e^{-8K} = P_2(K) \tag{4}$$

$$\Lambda e^{-2\tilde{L}+\tilde{M}} = 3e^{4K} + 4e^{2K} + 2 + 4e^{-2K} + 3e^{-4K} = P_3(K) \tag{5}$$

$$\Lambda e^{-\tilde{M}} = e^{6K} + e^{4K} + 3e^{2K} + 6 + 3e^{-2K} + e^{-4K} + e^{-6K} = P_4(K). \tag{6}$$

The results of this transformation is to give an Ising model on the A spins, with nearest-neighbour coupling $2\tilde{K}$, a diagonal next-nearest-neighbour coupling $\tilde{L}$ and a four-spin coupling $\tilde{M}$ (see figure 2(*b*)). Transformations of this type, and other types, have a long history in Ising model studies (Fisher 1959, Syozi 1972).

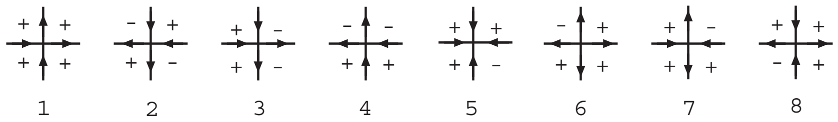

Figure 3. The eight spin/vertex configurations. Reversal of all spins corresponds to the same vertex.

It was shown by Wu (1971) that an Ising model of this type could be transformed to an 8-vertex model in an external electric field. This results in the model shown in figure 3, with Boltzmann weights ω_i given by

$$\omega_1 = P_1(K) \qquad \omega_2 = P_2(K) \qquad \omega_3 = \omega_4 = P_3(K) \qquad \omega_5 = \omega_6 = \omega_7 = \omega_8 = P_4(K). \tag{7}$$

Fan and Wu (1970) have shown that, provided the vertex weights satisfy the 'free-fermion' condition

$$\omega_1\omega_2 + \omega_3\omega_4 = \omega_5\omega_6 + \omega_7\omega_8, \tag{8}$$

the model has the same critical behaviour as the solvable nearest-neighbour Ising model. Furthermore the critical point is given by the condition

$$\omega_1 + \omega_2 + \omega_3 + \omega_4 = 2\max\{\omega_1, \omega_2, \omega_3, \omega_4\}. \tag{9}$$

In our case condition (8) becomes

$$P_1 P_2 + P_3^2 = 2P_4^2 \tag{10}$$

which is satisfied identically by the expressions (3)–(6). Thus our model falls within the standard Ising universality class. It is easily seen that ω_1 is always the maximum weight, and hence the criticality condition is

$$P_1 = P_2 + 2P_3. \tag{11}$$

Writing equation (11) in terms of the variable $y = \mathrm{e}^{2K}$ gives, after factorization,

$$y^6 - 2y^5 - 5y^4 + 4y^3 - 7y^2 + 2y - 1 = 0 \tag{12}$$

with the physical root $y = 3.3203$, giving the critical coupling

$$K_c = 0.600\,03\ldots. \tag{13}$$

In terms of the variable $t = \tanh K$ equation (12) takes the slightly simpler form

$$2t^6 - 4t^5 - 3t^4 - 4t^3 + 1 = 0. \tag{14}$$

We turn now to the magnetization per spin, which has the form

$$m \equiv M/N = \langle\sigma\rangle + 4\langle s\rangle \tag{15}$$

where we denote, as before, the spins on sublattices A and B by σ, s respectively. The expectation $\langle\sigma\rangle$ can be obtained directly from the vertex model, following Baxter (1986). The expression is

$$\langle\sigma\rangle = (1 - \Omega^{-2})^{1/8} \tag{16}$$

with

$$\Omega^2 = (\Gamma^2 + h^2 - 1)/\Gamma^2 \tag{17}$$

and

$$\Gamma = 2(\omega_5\omega_6\omega_7\omega_8)^{1/2}/(\omega_1\omega_4 + \omega_2\omega_3) \tag{18}$$

$$h = \tfrac{1}{2}(\omega_2^2 + \omega_3^2 - \omega_1^2 - \omega_4^2)/(\omega_1\omega_4 + \omega_2\omega_3). \tag{19}$$

In the present case, a little algebra gives

$$\Omega = \frac{P_1^2 + P_2^2 - 2P_3^2}{2(P_1P_2 + P_3^2)} = \frac{(y+1)^2(y-1)^4(y^2+1)(y^4-2y^3+6y^2-2y+1)}{4y^2(y^4-y^3+4y^2-y+1)^2}$$
$$= \frac{8t^4(1+t^2)(1+3t^4)}{(1+t^4-2t^6)^2}. \tag{20}$$

Calculation of the B-site magnetization is a little more involved. We follow essentially the steps in Choy and Baxter (1987). Returning to figure 2(a), we can write

$$4\langle s\rangle = \langle s_1+s_2+s_3+s_4\rangle = \frac{\sum_{\{s,\sigma\}}(s_1+s_2+s_3+s_4)\mathrm{e}^{-\beta H(s,\sigma)}}{\sum_{\{s,\sigma\}}\mathrm{e}^{-\beta H(s,\sigma)}} = \frac{\Lambda^{N-1}\sum_{\{\sigma\}}\mathrm{e}^{-\beta\tilde H}X}{\Lambda^N\sum_{\{\sigma\}}\mathrm{e}^{-\beta\tilde H}} \tag{21}$$

where summation over s-spins has been carried out completely in the denominator, and over all A plaquettes except one in the numerator. $\tilde H$ is the transformed Hamiltonian. The factor X is

$$X = \exp[-\tilde K(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_4+\sigma_1\sigma_4)]\exp[-\tilde L(\sigma_1\sigma_3+\sigma_2\sigma_4)]\exp[-\tilde M\sigma_1\sigma_2\sigma_3\sigma_4]$$
$$\times\sum_{s_1,s_2,s_3,s_4}(s_1+s_2+s_3+s_4)\exp[K(\sigma_1s_1+\sigma_2s_2+\sigma_3s_3+\sigma_4s_4)]$$
$$\times\exp[K(s_1s_2+s_2s_3+s_3s_4+s_1s_4)].$$

This can be evaluated, after some algebra, to give

$$X = X_1(\sigma_1+\sigma_2+\sigma_3+\sigma_4) + X_2(\sigma_1\sigma_2\sigma_3+\sigma_1\sigma_2\sigma_4+\sigma_1\sigma_3\sigma_4+\sigma_2\sigma_3\sigma_4) \tag{22}$$

with

$$X_1 = \Lambda\frac{t(1+t)(1+t^2)(1-t+2t^2+4t^4)}{(1+t^2+2t^4)(1-2t+3t^2+2t^4)} \tag{23}$$

$$X_2 = -2\Lambda\frac{t^5(1+t)^2}{(1+t^2+2t^4)(1-2t+3t^2+2t^4)}. \tag{24}$$

Hence we obtain

$$\langle s\rangle = \frac{t(1+t)}{(1+t^2+2t^4)(1-2t+3t^2+2t^4)}\{(1+t^2)(1-t+2t^2+4t^4)\langle\sigma_1\rangle - 2t^4(1+t)\langle\sigma_1\sigma_2\sigma_3\rangle\} \tag{25}$$

an expression involving both $\langle\sigma\rangle$, which we already have, and a three-spin correlation $\langle\sigma_1\sigma_2\sigma_3\rangle$. This result is the analogue of equation (6) of Choy and Baxter (1987).

To compute the three-spin correlation one needs to use the equivalence of the free-fermion vertex model to a 'checkerboard' Ising model (Baxter 1986). Using the transformation equations therein, and equation (7) of Choy and Baxter, gives

$$\langle\sigma_1\sigma_2\sigma_3\rangle = \frac{F(t)}{G(t)}\langle\sigma\rangle \tag{26}$$

where

$$F(t) = 1 - \frac{t(1-t+t^2+t^3+2t^4)\Delta}{1-2t+3t^3+2t^4} - \frac{t^2(1+5t^4+2t^6)\Delta}{1+2t^2+5t^4+4t^6+4t^8} \tag{27}$$

$$G(t) = \frac{t^2(1-t)^2\Delta}{1-2t+3t^3+2t^4} - \frac{t^2(1+5t^4+2t^6)\Delta}{1+2t^2+5t^4+4t^6+4t^8} \tag{28}$$

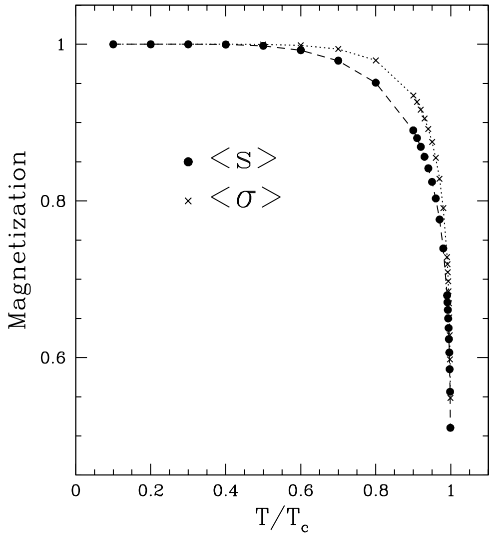

Figure 4. The A and B sublattice magnetizations versus T/T_c.

and

$$\Delta = \sqrt{2(1+t^2)/(1+3t^4)}. \tag{29}$$

Finally

$$\langle s\rangle = \frac{t(1+t)}{(1+t^2+2t^4)(1-2t+3t^3+2t^4)}\left\{(1+t^2)(1-t+2t^2+4t^4) - 2t^4(1+t)\frac{F(t)}{G(t)}\right\}\langle\sigma\rangle. \tag{30}$$

Equations (16), (20) and (30) provide a closed-form expression for the magnetization of both A and B sites on the 4–6 lattice.

These expressions can easily be evaluated numerically and in figure 4 we show the magnetizations as functions of temperature.

A partial check on the correctness of our results can be made by expanding the expressions in low-temperature series in the variable $u = e^{-2K}$. This yields

$$\langle\sigma\rangle = 1 - 2u^4 - 8u^5 - 36u^6 - 96u^7 - 252u^8 - 592u^9 - 1724u^{10} + \cdots \tag{31}$$

and

$$\langle s\rangle = 1 - 2u^3 - 6u^4 - 8u^5 - 16u^6 - 54u^7 - 240u^8 - 696u^9 - 1812u^{10} + \cdots. \tag{32}$$

These can be compared with low-temperature series derived directly (Domb 1973). To this order the agreement is perfect.

It is straightforward, in principle, to repeat this analysis for the case of two coupling constants K_1, K_2.

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